Chapter 14 Differentiation 2

1. The tangent to the curve $y = ln(3x^2 - 4) - \frac{x^3}{6}$, at the point where $x = 2$, meets the y-axis at the point P. Find the exact coordinates of P

$x = 0$

[6]

$$y = \ln(3x^2 - 4) - \frac{x^3}{6}$$

$$\frac{dy}{dx} = \frac{1}{3x^2 - 4} \times 6x - \frac{3x^2}{6}$$

$$= \frac{6x}{3x^2 - 4} - \frac{x^2}{2}$$

$$x = 2, \frac{dy}{dx} = \frac{12}{12 - 4} - \frac{4}{2}$$

$$= \frac{12}{8} - 2$$

$$= \frac{3}{2} - 2$$

$$= -\frac{1}{2}$$

$$y = \ln(3x^2 - 4) - \frac{8}{6}$$

$$= \ln 8 - \frac{4}{3}$$

$$y = -\frac{1}{2}x + c$$

$$\ln 8 - \frac{4}{3} = -1 + c$$

$$c = \ln 8 - \frac{1}{3}$$

$$\therefore y = -\frac{1}{2}x + \ln 8 - \frac{1}{3}$$

$$y = -\frac{1}{2}(0) + \ln 8 - \frac{1}{3}$$

$$y = \ln 8 - \frac{1}{3} \qquad \therefore P\left(0, \ln 8 - \frac{1}{3}\right)$$

2. Variables *x* and *y* are such that $y = \frac{e^{3x}sinx}{x^2}$. Use differentiation to find the approximate change in *y* as *x* increases from 0.5 to 0.5 + *h*, where *h* is small.

[6]

$y = \frac{e^{3x}\sin x}{x^2}$ $\quad x = 0.5 \quad \delta x = h \quad \delta y = ?$

$u = e^{3x}\sin x$

$u' = 3e^{3x}\sin x + e^{3x}\cos x$

$v = x^2$

$v' = 2x$

$\frac{dy}{dx} = \frac{u'v - v'u}{v^2} = \frac{(3e^{3x}\sin x + e^{3x}\cos x)x^2 - 2x(e^{3x}\sin x)}{x^4}$

$= \frac{(3e^{3/2}\sin\frac{1}{2} + e^{3/2}\cos\frac{1}{2})\frac{1}{4} - e^{3/2}\sin\frac{1}{2}}{(0.5)^4}$

$= \frac{(6.446 + 3.933)\frac{1}{4} - 2.149}{0.0625}$

$= 7.14$

$\frac{dy}{dx} \approx \frac{\delta y}{\delta x}$

$7.14 \approx \frac{\delta y}{h}$

$\delta y \approx 7.14h$

3. Find the equation of the tangent to the curve $y = \frac{ln(3x^2 - 1)}{x+2}$ at the point where $x = 1$. Give your answer in the form *y* = *mx*+*c*, where *m* and *c* are constants correct to 3 decimal places.

[6]

$$y = \frac{\ln(3x^2-1)}{x+2}$$

$$u = \ln(3x^2-1)$$

$$u' = \frac{1}{3x^2-1} \times 6x = \frac{6x}{3x^2-1}$$

$$v = x+2$$

$$v' = 1$$

$$\frac{dy}{dx} = \frac{u'v - v'u}{v^2}$$

$$= \frac{\frac{6x}{3x^2-1} \times (x+2) - \ln(3x^2-1)}{(x+2)}$$

$$x=1, \quad \frac{dy}{dx} = \frac{\frac{6}{3-1} \times 3 - \ln 2}{3}$$

$$= 0.923$$

$$x=1, \quad y = \frac{\ln(3x^2-1)}{x+2}$$

$$= \frac{\ln 2}{3}$$

$$y = mx + c$$

$$\frac{\ln 2}{3} = 0.923 + c$$

$$c = -0.692$$

$$\therefore y = 0.923x - 0.692$$

4. (a) Find the x-coordinates of the stationary points of the curve $y = e^{3x}(2x + 3)^6$.

$y = e^{3x}(2x+3)^6$

$u = e^{3x}$

$u' = 3e^{3x}$

$v = (2x+3)^6$

$v' = 6(2x+3)^5 \times 2$

$= 12(2x+3)^5$

[6]

$\frac{dy}{dx} = u'v + v'u$

$0 = 3e^{3x}(2x+3)^6 + 12(2x+3)^5 \times e^{3x}$

$0 = 3e^{3x}(2x+3)^5\left[(2x+3)+4\right]$

$0 = 3e^{3x}(2x+3)^5(2x+7)$

$3e^{3x} = 0$ $\quad$ $2x+3=0$ $\quad$ $2x+7=0$

$e^{3x} = 0$ $\quad$ $x = -\frac{3}{2}$ $\quad$ $x = -\frac{7}{2}$

(reject)

(b) A curve has equation $y = f(x)$ and has exactly two stationary points. Given that $f''(x) = 4x - 7$, $f'(0.5) = 0$ and $f'(3) = 0$, use the second derivative test to determine the nature of each of the stationary points of this curve.

[2]

$x = 0.5$, $f''(0.5) = 2 - 7$

$= -5 < 0$ maximum pt

$x = 3$, $f''(3) = 12 - 7$

$= 5 > 0$ minimum pt

5. Variables *x* and *y* are such that $y = \sin x + e^{-x}$. Use differentiation to find the approximate change in *y* as *x* increases from $\frac{\pi}{4}$ to $\frac{\pi}{4} + h$, where *h* is small. [4]

$$\frac{dy}{dx} = \cos x - e^{-x}$$

$x = \frac{\pi}{4}$, $\delta x = h$

$\delta y = ?$

$$\frac{dy}{dx} = \frac{\sqrt{2}}{2} - e^{-\pi/4}$$
$$= 0.251$$

$$\frac{dy}{dx} \approx \frac{\delta y}{\delta x}$$

$$0.251 \approx \frac{\delta y}{h}$$

$$\delta y \approx 0.251h$$

6. (a) Find the equation of the tangent to the curve $2y = \tan 2x + 7$ at the point where $x = \frac{\pi}{8}$. Give your answer in the form $ax - y = \frac{\pi}{b} + c$, where *a*, *b* and *c* are integers [5]

$$y = \frac{1}{2}\tan 2x + \frac{7}{2}$$

$x = \frac{\pi}{8}$, $y = \frac{1}{2}\tan\frac{\pi}{4} + \frac{7}{2}$
$= 4$

$$\frac{dy}{dx} = \frac{1}{\cancel{2}}\sec^2 2x \times \cancel{2}$$
$$= \sec^2 2x$$

$$x = \frac{\pi}{8},\ \frac{dy}{dx} = \sec^2\frac{\pi}{4}$$
$$= \frac{1}{\cos^2\frac{\pi}{4}} = 2$$

$$y = 2x + c$$
$$4 = \frac{\pi}{4} + c$$
$$c = 4 - \frac{\pi}{4}$$

$$y = 2x + 4 - \frac{\pi}{4}$$

$$\frac{\pi}{4} - 4 = 2x - y$$

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$y=0$ $\quad$ $x=0$

(b) This tangent intersects the x-axis at P and the y-axis at Q. Find the length of PQ. [2]

$$\frac{\pi}{4}-4 = 2x-y$$
$$\frac{\pi}{4}-4 = 2x$$
$$x = \frac{\pi}{8}-2$$
$$P\left(\frac{\pi}{8}-2, 0\right)$$

$$\frac{\pi}{4}-4 = 2x-y$$
$$\frac{\pi}{4}-4 = y$$
$$Q\left(0, \frac{\pi}{4}-4\right)$$

$$PQ = \sqrt{\left(\frac{\pi}{4}-4\right)^2+\left(\frac{\pi}{8}-2\right)^2}$$
$$= 3.59$$

7. (a) Differentiate $y = tan\,(x+4) - 3\,sin\,x$ with respect to x. [2]

$$\frac{dy}{dx} = \sec^2(x+4) - 3\cos x$$

(b) Variables x and y are such that $y = \frac{ln\,(2x+5)}{2e^{3x}}$. Use differentiation to find the approximate change in y as x increases from 1 to 1 + h, where h is small. [6]

$$u = \ln(2x+5) \qquad v = 2e^{3x}$$
$$u' = \frac{1}{2x+5}\times 2 \qquad v' = 6e^{3x}$$

$$y' = \frac{u'v - v'u}{v^2} = \frac{\frac{2}{2x+5}\times 2e^{3x} - 6e^{3x}\times\ln(2x+5)}{4e^{6x}}$$

$$= \frac{\frac{4e^{3x}}{2x+5} - 6e^{3x}\ln(2x+5)}{4e^{6x}}$$

$$x=1,\ y' = -0.138$$

$$\frac{dy}{dx} \approx \frac{\delta y}{\delta x}$$

$$-0.138 \approx \frac{\delta y}{h}$$

$$\delta y \approx -0.138h$$

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8. It is given that $y = \frac{\tan 3x}{\sin x}$.

a. Find the exact value of $\frac{dy}{dx}$ when $x = \frac{\pi}{3}$. [4]

$u = \tan 3x \qquad v = \sin x$

$u' = 3\sec^2 3x \qquad v' = \cos x$

$$\frac{dy}{dx} = \frac{u'v - v'u}{v^2} = \frac{3\sec^2 3x \sin x - \cos x \tan 3x}{\sin^2 x}$$

$$= \frac{\frac{3}{\cos^2 \pi} \times \sin\frac{\pi}{3} - \cos\frac{\pi}{3}\tan\pi}{\sin^2\frac{\pi}{3}}$$

$$= \frac{3 \times \frac{\sqrt{3}}{2}}{\frac{3}{4}} = \frac{\cancel{3}\sqrt{3}}{\cancel{2}_1} \times \frac{\cancel{4}^2}{\cancel{3}} = 2\sqrt{3}$$

b. Hence find the approximate change in y as x increases from $\frac{\pi}{3}$ to $\frac{\pi}{3} + h$, where h is small. [1]

$x = \frac{\pi}{3}, \ \delta x = h, \ \delta y = ?$

$\frac{dy}{dx} \approx \frac{\delta y}{\delta x}$

$2\sqrt{3} \approx \frac{\delta y}{h}$

$\delta y \approx 2\sqrt{3}h$

c. Given that x is increasing at the rate of 3 units per second, find the corresponding rate of change in y when $x = \frac{\pi}{3}$, giving your answer in its simplest surd form. [2]

$\frac{dy}{dt} = ? \qquad \frac{dx}{dt} = 3, \ x = \frac{\pi}{3}$

$$\frac{dy}{dt} = \frac{dy}{dx} \times \frac{dx}{dt}$$

$$= 2\sqrt{3} \times 3$$

$$= 6\sqrt{3}$$

9. A curve has equation $y = \frac{ln(3x^2-5)}{2x+1}$ for $3x^2 > 5$.

a. Find the equation of the normal to the curve at the point where $x = \sqrt{2}$. [6]

$\frac{dy}{dx} = \frac{u'v - v'u}{v^2}$

$u = \ln(3x^2-5)$

$u' = \frac{1}{3x^2-5} \times 6x$

$v = 2x+1$

$v' = 2$

$= \frac{\frac{6x}{3x^2-5} \times (2x+1) - 2\ln(3x^2-5)}{(2x+1)^2}$

$x = \sqrt{2}$, $\frac{dy}{dx} = \frac{24-6\sqrt{2}}{7}$

normal $\Rightarrow m_2 = \frac{-7}{24-6\sqrt{2}} = -0.45$

$x = \sqrt{2}$, $y = \frac{\ln(3x^2-5)}{2x+1} = \frac{\ln 1}{2x+1} = 0$

$y = -0.45x + c$

$0 = -0.636 + c$

$c = 0.636$

$\therefore y = -0.45x + 0.64$

b. Find the approximate change in y as x increases from $\sqrt{2}$ to $\sqrt{2}$ + h, where h is small. [1]

$\delta y = ?$, $x = \sqrt{2}$, $\delta x = h$

$\frac{dy}{dx} \approx \frac{\delta y}{\delta x}$

$2.22 \approx \frac{\delta y}{h}$

$\delta y \approx 2.22h$

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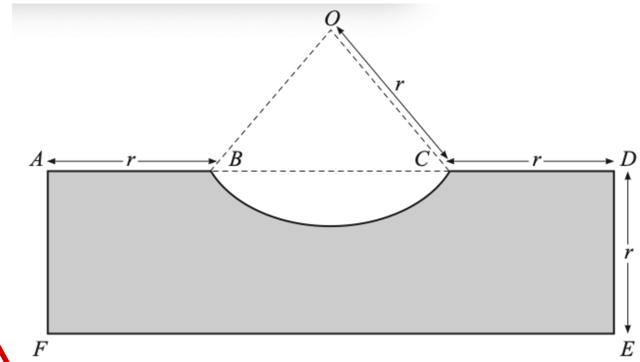
10. In this question all lengths are in centimetres and all angles are in radians.



The diagram shows the rectangle *ADEF*, where *AF* = *DE* = *r* . The points *B* and *C* lie on *AD* such that *AB* = *CD* = *r*. The curve *BC* is an arc of the circle, centre *O*, radius *r* and has a length of 1.5*r*.

a. Show that the perimeter of the shaded region is (7.5 + 2 sin 0.75) *r* .

[5]

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b. Find the area of the shaded region, giving your answer in the form kr^2, where k is a constant correct to 2 decimal places.

[4]

11. (a) Given that $y = \frac{e^{2x-3}}{x^2+1}$, find $\frac{dy}{dx}$. [3]

$u = e^{2x-3}$ $\quad v = x^2+1$

$u' = 2e^{2x-3}$ $\quad v' = 2x$

$$\frac{dy}{dx} = \frac{u'v - v'u}{v^2} = \frac{2e^{2x-3} \times (x^2+1) - 2x(e^{2x-3})}{(x^2+1)^2}$$

(b) Hence, given that *y* is increasing at the rate of 2 units per second, find the exact rate of change of *x* when *x* = 2. [3]

$\frac{dy}{dt} = 2$, $x = 2$, $\frac{dx}{dt} = ?$

$$\frac{dy}{dx} = \frac{2e^{2x-3} \times (x^2+1) - 2x(e^{2x-3})}{(x^2+1)^2}$$

$$= \frac{2e \times 5 - 4e}{25} = \frac{6e}{25}$$

$$\frac{dx}{dt} = \frac{dx}{dy} \times \frac{dy}{dt}$$

$$= \frac{25}{6e} \times 2 = \frac{25}{3e}$$

12. It is given that $y = ln(sin\,x + 3cos\,x)$ for $0 < x < \frac{\pi}{2}$.

a. Find $\frac{dy}{dx}$.

$$\frac{dy}{dx} = \frac{1}{\sin x + 3\cos x} \times \cos x - 3\sin x$$

[3]

$$= \frac{\cos x - 3\sin x}{\sin x + 3\cos x}$$

b. Find the value of x for which $\frac{dy}{dx} = -\frac{1}{2}$.

$$\frac{\cos x - 3\sin x}{\sin x + 3\cos x} = -\frac{1}{2}$$

[3]

$$2\cos x - 6\sin x = -\sin x - 3\cos x$$

$$\cancel{5}\cos x = \cancel{5}\sin x$$

$$1 = \tan x$$

$$x = \tan^{-1}(1)$$

$$= \frac{\pi}{4}$$

13. It is given that $y = ln(1 + sin\,x)$ for $0 < x < \pi$.

a. Find $\frac{dy}{dx}$.

$$\frac{dy}{dx} = \frac{1}{1+\sin x} \times \cos x$$

[2]

$$= \frac{\cos x}{1+\sin x}$$

b. Find the value of $\frac{dy}{dx}$ when $x = \frac{\pi}{6}$, giving your answer in the form $\frac{1}{\sqrt{a}}$, where *a* is an integer.

$$\frac{dy}{dx} = \frac{\cos x}{1+\sin x}$$

[2]

$$x = \frac{\pi}{6},\ \frac{dy}{dx} = \frac{\sqrt{3}}{2} \div \frac{3}{2}$$

$$= \frac{\sqrt{3}}{2} \times \frac{2}{3} = \frac{\sqrt{3}}{3} = \frac{3}{3\sqrt{3}} = \frac{1}{\sqrt{3}}$$

c. Find the value of x for which $\frac{dy}{dx} = tan\,x$.

$$\frac{\cos x}{1+\sin x} = \tan x$$

[5]

$$\cos x = \tan x + \tan x \sin x$$

$$\cos x = \frac{\sin x}{\cos x} + \frac{\sin^2 x}{\cos x}$$

$$\cos^2 x = \sin^2 x + \sin x$$

$$(1 - \sin^2 x) = \sin^2 x + \sin x$$

$$0 = 2\sin^2 x + \sin x - 1$$

$$\sin x = \frac{1}{2} \quad \text{or} \quad \sin x = -1$$

(reject)

$$x = \sin^{-1}\left(\frac{1}{2}\right)$$

$$= \frac{\pi}{6}, \frac{5\pi}{6}$$